Zero Forcing with Random Sets

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Joint with Bryan Curtis, Luyining Gan, Jamie Haddock, and Rachel Lawrence

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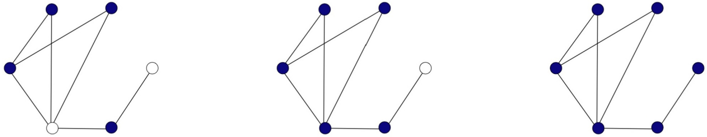

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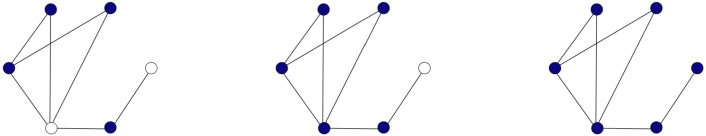

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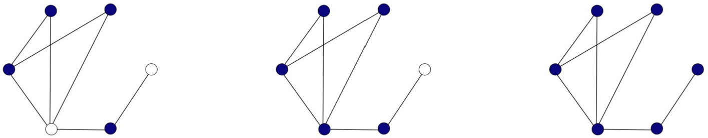

We say that $B \subseteq V(G)$ is a *zero forcing set* if this process ends with every vertex colored blue, and we let $\mathrm{zfs}(G)$ be the set of zero forcing sets. Define the zero forcing number $Z(G) := \min_{B \in \mathrm{zfs}(G)} |B|$.

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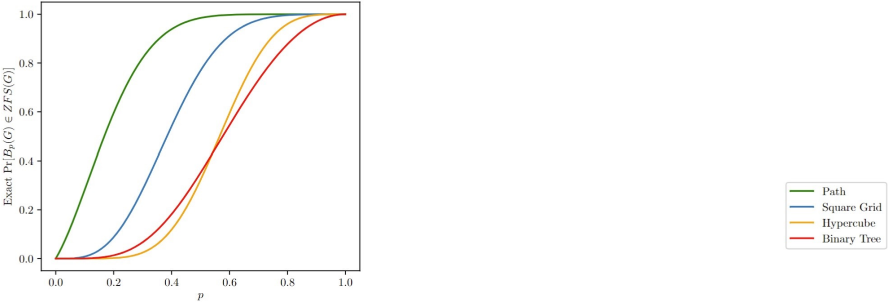
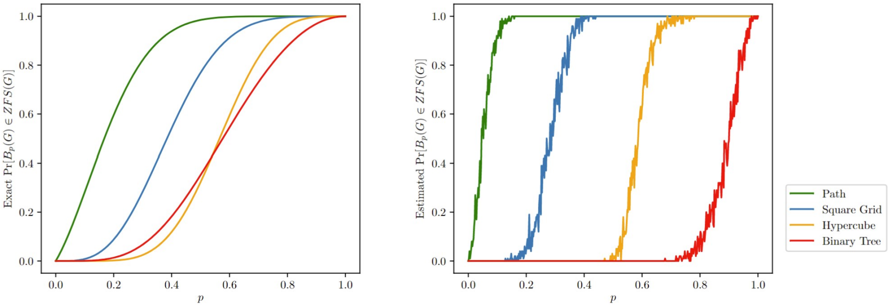
Problem

Determine or bound $\Pr[B_p(G) \in \mathrm{zfs}(G)]$.



Exact $\Pr[B_p(G) \in ZFS(G)]$
0.0
0.2
0.4
0.6
0.8
1.0
p
Path
Square Grid
Hypercube
Binary Tree



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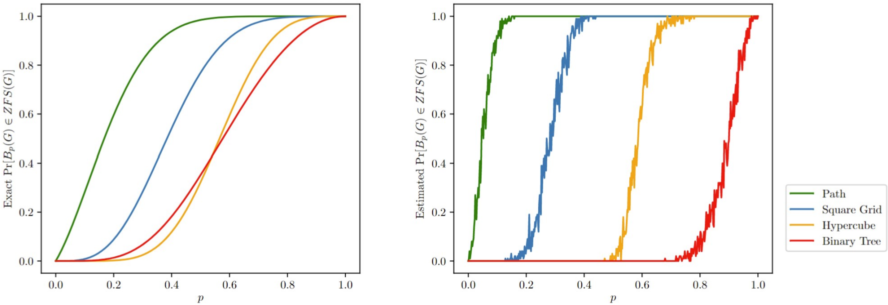

Define the threshold probability $p(G)$ to be the unique p such that $\Pr[B_p(G) \in \text{zfs}(G)] = 1/2$.

Family	Threshold Probability
K_n	$1 - \Theta(n^{-1})$
nK_1	$2^{-1/n}$
$K_{n_1,\cdots,n_k}$	$1 - \Theta_k(\min_i\{n_i^{-1}\})$
P_n	$\Theta(n^{-1/2})$
C_n	$\Theta(n^{-1/2})$
W_n	$\Theta(n^{-1/3})$

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Corollary (Informal)

For every n-vertex graph G, a random set of size much less than $\sqrt{n}$ is unlikely to be a zero forcing set.

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Proposition

If G is an n-vertex graph, then

$$\Pr[B_p(G) \in \mathrm{zfs}(G)] \geq \Pr[B_p(\overline{K_n}) \in \mathrm{zfs}(\overline{K_n})],$$

with equality if and only if $p \in \{0, 1\}$ or $G = \overline{K_n}$.

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This is a weaker version of a conjecture of Boyer et. al. which says for all k

$$|\{B \in \mathrm{zfs}(G) : |B| = k\}| \leq |\{B \in \mathrm{zfs}(P_n) : |P_n| = k\}|.$$

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Theorem (CGHLS 2022)

There exists some $n_0 \in \mathbb{N}$ such that if T is an n-vertex tree with $n \geq n_0$, then

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If $p \geq e^{-1/\delta}$ then the result is trivial, and otherwise each term is minimized when $\deg(v) \geq \delta$ is as small as possible. □

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and for this to be at least $1/2$ we need $p = \Omega(n^{-1/2})$. □

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Theorem (CGHLS 2022)

If T is an n-vertex tree with $n \geq n_0$, then

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If $p \ll n^{-1}$, we give a crude upper bound for the number of zero forcing sets of size k which is significantly better than the count for the path when k is small. Since $B_p(T)$ will be very small, this gives the result. □

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Question

Given G, p, k, which set $B \subseteq V(G)$ of size k maximizes $\Pr[B_p \in \mathrm{zfs}(G)]$? What is this value?